Rotating Electrically Conducting Fluids in a Long Cylinder

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A hydromagnetically generated vortex in an incompressible, viscous, cylindrically confined plasma is analyzed Solutions for steady laminar one- and two-dimensional flow fields are obtained The two-dimensional flow field involves impressed radial mass flow The solutions for the pressure of the flow field with impressed radial mass flow are valid for small magnetic Reynold's number only Results are given which have application to a gaseous nuclear rocket

Nomenclature

B = magnetic induction
B_0 = applied axial magnetic field
E = electric field intensity
G_0 = radial mass flow rate (mass/area-sec)
h_0 = half-length of cylinder
I = total radial electric current
J = electric current density
p = pressure
P_m = electric power input per unit mass
Q_j = Joule heating
Q = viscous dissipation rate
r,θ,z = radial, azimuthal, and axial positions
V = velocity
u,V_θ,w = velocity components in r, θ, and z directions
ϕ_0 = electrical potential difference applied across the electrodes
ρ = density
η = absolute coefficient of viscosity
σ = electrical conductivity
μ = magnetic permeability

Subscripts

0 = a quantity at the outer wall or a constant quantity
1 = a quantity at the inner wall
r,θ,z = component quantity along coordinate axis

Dimensionless variables and parameters

b = magnetic induction
e = electric field intensity
j = electric current density
u = radial velocity
v = azimuthal velocity
π = pressure
L = h_0/r_0 = ratio of half length to outer radius
ξ = z/r_0 = axial position
ζ = r/r_0 = radial position
ζ_1 = r_1/r_0 = ratio of inner radius to outer radius
M = $B_0r_0(\sigma/\eta)^{1/2}$ = Hartmann number
R_N = G_0r_0/η = radial Reynolds number
m = $R_N + 1$ =
β_1 = $\dfrac{1}{2\pi}\dfrac{1/2\sigma h_0}{\phi_0/I}$ = $\dfrac{1}{2\pi}\dfrac{\text{resistance of plasma}}{\text{electrical resistance}}$
β_2 = $\dfrac{I_rB_0/4\pi r_0h_0}{G_0\phi_0/B_0r_0^2}$ = $\dfrac{\text{azimuthal Lorentz force}}{\text{drift momentum rate per unit volume}}$
β = $\dfrac{G_0/\rho}{\phi_0/B_0r_0}$ = $\dfrac{\text{radial velocity at the outer wall}}{\text{drift velocity}}$
R_u = $\mu\sigma G_0r_0/\rho$ = radial magnetic Reynolds number
λ_2 = $h_0\eta/\rho\phi_0$

Abbreviations

ZRMF = zero radial mass flux ($u = 0$)
NRMF = nonzero radial mass flux ($u \neq 0$)

1 Introduction

CONSIDER an electrically conducting fluid in an annular region between two coaxial cylinders, having radii r_0 and r_1, whose circumferential walls are perfectly conducting electrodes (see Fig 1) A uniform magnetic field B_0 is applied axially while an electric potential difference of ϕ_0 is maintained across the electrodes As a result of the radial electric current, J flowing per unit area normal to the magnetic field, a Lorentz force sets the fluid in rotation Consequently, the fluid rotates with the azimuthal velocity V_θ Associated with this rotation is a centrifugal force that sets up a radial pressure gradient

Chang and Lundgren[1] have studied this flow field and have obtained solutions for the steady and transient velocity fields, electric currents, etc for the special case where the radius is large compared to the length of the cylinder Lewellen[2] has also studied such flow fields by an approximation method that is based on expansion of all the quantities in terms of a power series of the magnetic Reynolds number A relevant experimental investigation has been reported by Anderson et al ,[3] and a recent paper, giving some experimental results on the boundary effects in viscous rotating plasmas, has been presented by Kunkel et al [4] A review of high-temperature rotating plasma experiments, primarily concerned with thermonuclear research, has also been given by Wilcox [5]

This paper is concerned with the laminar flow of an incompressible viscous fluid in a coaxial cylindrical geometry where the axial dimensions are much larger than radial dimensions In addition, the effect of an impressed radial mass flow through the boundary walls is examined This is motivated by the application of the rotating plasma to a gaseous nuclear rocket, which is described and analyzed by Gross and Kessey [6]

2 Differential Equations and Boundary Conditions

The conventional nonrelativistic magnetohydrodynamic equations will be used [7,8] For simplicity it is assumed that 1) the plasma is incompressible, 2) the Lorentz force is the only body force, and 3) the permeability and electrical conductivity are scalar constant quantities

For the physical problem, it is further assumed that 4) the flow is axially symmetrical, 5) the axial component of velocity w is negligibly small compared to the remaining components, 6) the azimuthal current J_θ is negligibly small, and 7) axial viscous effects on the velocity field are negligibly small compared to radial viscous effects

Assumption 5 implies that secondary flows are neglected The importance of secondary flows in such a long cylinder is unknown; it is suspected to be important and requires further

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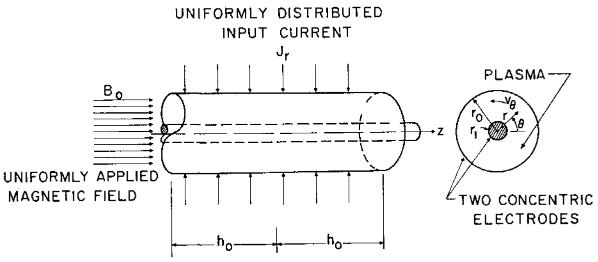

Fig 1 Model and coordinates

investigation Such an investigation will usually start as a perturbation to the flow fields described by the present theory The neglect of azimuthal current J_θ is valid when the absolute value of the radial magnetic Reynolds number is much less than unity, i e , $|R_u| \ll 1$ or the radial Reynolds is small † Assumption 7 implies that the stress tensor due to an axial shear force acting on an element of area in the θ direction is approximately zero and, consequently, for an incompressible viscous fluid, $\partial V_\theta/\partial z = 0$ This is a direct consequence of the assumption of the long cylinder Based on assumptions 4–6 it can be shown[9] that $B = 0$ and $B = B_0$, the uniform applied magnetic field Therefore, with these assumptions, the relevant equations for a steady-state‡ rotating plasma with a radial mass flow are

$$\frac{\partial p}{\partial r} = -\rho\left[u\frac{\partial u}{\partial r} - \frac{V_\theta^2}{r}\right] - \frac{B_\theta}{\mu r}\frac{\partial}{\partial r}(rB_\theta) \tag{1}$$

$$\frac{\partial p}{\partial z} = -\frac{1}{\mu}\frac{\partial}{\partial z}\left(\frac{B_\theta^2}{2}\right) \tag{2}$$

$$E = (J/\sigma) - V_\theta B_0 \tag{3}$$

$$\rho u\left[\frac{dV_\theta}{dr} + \frac{V_\theta}{r}\right] = \eta\left\{\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rV_\theta)\right]\right\} - B_0 J_r \tag{4}$$

$$\frac{1}{\mu\sigma}\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\right] - \mu\frac{\partial J_r}{\partial z}\right\} = -\frac{uB_\theta}{r} + u\frac{\partial B_\theta}{\partial r} \tag{5}$$

These equations can be simplified by noting that, from the conservation of mass and radial electric current,

$$u = (G_0 r_0/\rho)1/r \qquad J = I/4\pi h_0 r \tag{6}$$

in which G_0 is the mass of plasma injected at $r = r_0$ per unit of area and time, and I is the total radial electric current (constant if it is assumed that, for a long cylinder, $\partial I_r/\partial z = 0$) Therefore, Eqs (4) and (5) are second-order partial differential equations involving V_θ and B_θ only To solve them, two boundary conditions on V_θ and four on B_θ are required The requirement of no slip at $r = r_1$ and $r = r_0$ for the viscous fluid implies that $V_\theta = 0$ at $r = r_1$ and $r = r_0$

To determine the boundary conditions on B_θ, consider the three regions of the cylindrical structure, namely, the outside, region 1; the wall material, region 2; and the cylindrical annular volume, region 3 (Fig 2) If one assumes that there are no surface currents in this model, then E, which is continuous at the boundary surfaces, is zero both inside the perfectly conducting electrodes and on the circumferential walls Therefore, the z component of Ohm's law ($\mathbf{J} = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B}]$) and Ampere's law ($\nabla \times \mathbf{B} = \mu\mathbf{J}$) yields the following two boundary conditions on B_θ:

$$(\partial/\partial r)(rB_\theta) = \mu\sigma r u B_\theta \text{ at } r = r_1,\ r = r_0 \tag{7}$$

† I am grateful to W B Kunkel for calling my attention to the fact that neglecting azimuthal current effects on the radial pressure gradient (for $G_0 \neq 0$) implies a small radial magnetic Reynolds number $|R_u| \ll 1$

‡ Solutions for the transient velocity and pressure fields are reported in Ref 9

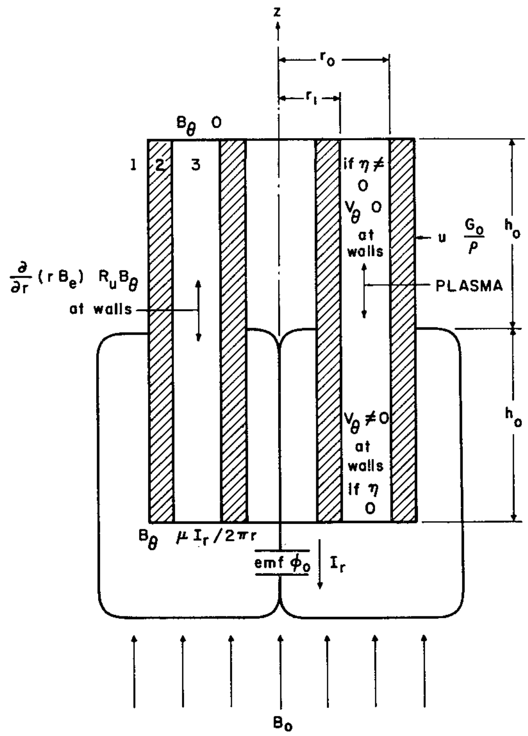

Fig 2 Boundary conditions of flow

Based on the same assumption, B_θ, which is the component of $\mathbf{B}$ tangent to the end faces, must be continuous at the insulated walls $z = -h_0$ and $z = h_0$ In the outside region the current is zero and, therefore, from Ampere's law,

$$(\partial/\partial z)(rB_\theta) = 0 \qquad \text{for } z > h_0 \quad z < -h_0$$

$$(\partial/\partial r)(rB_\theta) = 0 \qquad \text{for } z > h_0 \quad z < -h_0$$

The solution of the last two equations is $rB_\theta = \text{const}$ for $z > h_0$ and $z < -h_0$ For $z > h_0$, i e , the region exterior to the loop of the external electrical circuit, the constant is zero; for $z < -h_0$, i e , in the loop of the external electrical circuit, the constant is $I\ \mu/4\pi$ Hence, the remaining two boundary conditions on B_θ are

$$B_\theta = 0 \text{ for } z \geqq h_0 \tag{8}$$

$$B_\theta = \mu I/4\pi r \text{ for } z \leqq -h_0 \tag{9}$$

It is convenient to use these equations in dimensionless forms To obtain dimensionless equations, define dimensional constants as $E = \phi_0/r_0$, $V = \phi_0/B_0 r_0$, $I = 2\pi h_0\eta\phi_0/B_0^2 r_0^2$, $P = \rho\phi_0^2/2B_0^2 r_0$, and $B = \mu\rho(\phi_0/B_0 r_0)^2$ such that $E = Ee$, $V_\theta = Vv$, $I = Ii$, $p = P\pi$, and $B_\theta = Bb$ in which e, v, i, π, and b are, respectively, the dimensionless electric field, rotational velocity, radial electric current, pressure and azimuthal magnetic field Using these definitions, one obtains the following dimensionless differential equations:

$$\frac{1}{2}\frac{\partial\pi}{\partial\zeta} = \frac{\beta^2}{\zeta^3} + \frac{v^2}{\zeta} - \frac{b^2}{\zeta}\frac{\partial}{\partial\zeta}(\zeta b) \tag{10}$$

$$\frac{1}{2}\frac{\partial\pi}{\partial\xi} = -\frac{\partial}{\partial\xi}\left(\frac{b^2}{2}\right) \tag{11}$$

$$e = (i/2M^2\zeta) - v \tag{12}$$

$$R_N\left[\frac{1}{\zeta}\frac{dv}{d\zeta} + \frac{v}{\zeta^2}\right] - \frac{d}{d\zeta}\left[\frac{1}{\zeta}\frac{d}{d\zeta}(\zeta v)\right] = -\frac{i}{2\zeta} \tag{13}$$

$$\frac{\partial}{\partial\zeta}\left[\frac{1}{\zeta}\frac{\partial}{\partial\zeta}(\zeta b)\right] - R_u\left[\frac{1}{\zeta}\frac{\partial b}{\partial\zeta} - \frac{b}{\zeta^2}\right] = 0 \tag{14}$$

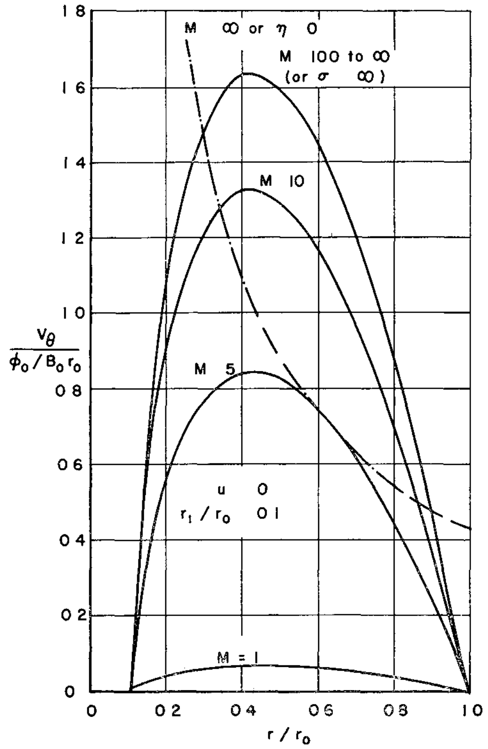

Fig 3 Variation of the rotational velocity of the viscous fluid with M ($u = 0$)

having the boundary conditions

$$v = 0 \text{ at } \zeta = \zeta_1 \qquad \zeta = 1 \tag{15}$$

$$(\partial/\partial\zeta)(\zeta b) = R_u b \text{ at } \zeta = \zeta_1 \qquad \zeta = 1 \tag{16}$$

$$b = 0 \text{ at } \xi = L \tag{17}$$

$$b = \lambda_2 i/2\zeta \text{ at } \xi = -L \tag{18}$$

Equations (13) and (14) are second-order partial differential equations involving the unknown quantities v and b, and their solutions require the boundary conditions set forth in Eqs (15–18) In terms of these solutions the remaining quantities of interest may be obtained For example, the pressure is obtained as follows:

Integrating Eq (11),

$$\pi/2 = -(b^2/2) + g(\zeta) \tag{19}$$

in which $g(\zeta)$ is a constant of integration Then differentiate Eq (19) with respect to ζ to get

$$\frac{1}{2}\frac{\partial\pi}{\partial\zeta} = -\frac{\partial}{\partial\zeta}\left(\frac{b^2}{2}\right) + \frac{dg}{d\zeta}$$

Combining the last equation with Eq (10), one finds that

$$\frac{dg}{d\zeta} = \frac{\beta^2}{\zeta^3} + \frac{v^2}{\zeta} - \frac{b^2}{\zeta}$$

Hence, integrating,

$$g(\zeta) = \int\left[\frac{\beta^2}{\zeta^3} + \frac{v^2}{\zeta} - \frac{b^2}{\zeta}\right]d\zeta + A_1$$

in which A_1 is a pure constant of integration Therefore, substituting for $g(\zeta)$ in Eq (19), one finally gets the pressure field

$$\frac{\pi}{2} = -\frac{b^2}{2} + \int\left[\frac{\beta^2}{\zeta^3} + \frac{v^2}{\zeta} - \frac{b^2}{\zeta}\right]d\zeta + A_1 \tag{20}$$

3 Solutions

From Eqs (13) and (14), the solutions for v and b will now be determined

For the one-dimensional viscous flow $\beta = R_N = R_u = \partial v/\partial\xi = 0$; therefore, Eqs (13) and (14) become [for the case of zero radial mass flux, $u = 0$ – (ZRMF)]

$$-\frac{V}{\zeta^2} + \frac{d^2v}{d\zeta^2} + \frac{1}{\zeta}\frac{dv}{d\zeta} = \frac{i}{2\zeta} \tag{21}$$

$$-\frac{b}{\zeta^2} + \frac{\partial^2 b}{\partial\zeta^2} + \frac{1}{\zeta}\frac{\partial b}{\partial\zeta} = 0 \tag{22}$$

The solution of Eq (22), which satisfies the boundary conditions expressed in Eq (16), is found to be $b = a_1(\xi)/\zeta$ where $a_1(\xi)$ is a function of ξ only Again, under the long cylinder approximation in which the total radial electric current is assumed to be constant, $a_1(\xi) = a_2 + a_3\xi$, since $di/d\xi = d^2b/d\xi^2 = 0$, where a_2 and a_3 are pure constants Hence, $b = (a_2 + a_3\xi)/\zeta$ To satisfy the boundary conditions of Eqs (17) and (18),

$$b = -(\lambda^2 i/4\zeta)[(\xi/L) - 1] \tag{23}$$

The inhomogeneous equation (21) may be solved by transforming from the variable ζ to α with the relationship $\zeta = e^\alpha$ or $\alpha = \ln\zeta$ Thus, Eq (21) transforms into

$$(d^2v/d\alpha^2) - v = ie^\alpha/2$$

whose solution, which satisfies the boundary conditions given in Eq (15), is

$$v = (i/4)\{c[(1/\zeta) - \zeta] + \zeta\ln\zeta\} \tag{24}$$

with $c = -[\zeta_1^2 \ln\zeta_1/(1 - \zeta_1^2)]$ To obtain solutions in terms of the applied voltage ϕ_0, the total electric current may be determined from Eq (12) in the following manner The potential difference between the inner and outer electiodes

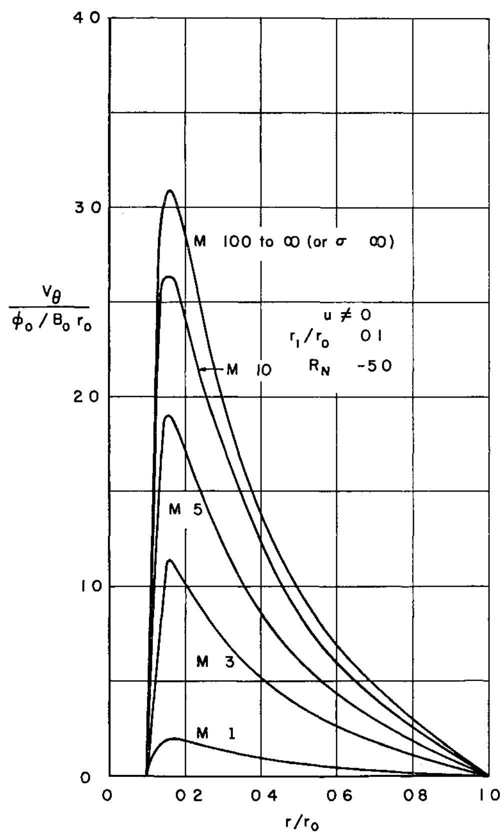

Fig 4 Variation of the rotational velocity of the viscous fluid with M ($u \neq 0$)

is found by integrating E between r_1 and r_0, namely,

$$\phi_0 = -\int_{r_1}^{r_0} E_r dr = -Er_0 \int_{\zeta_1}^{1} e d\zeta$$

having changed the variable of integration by the transformation $r = r_0\zeta$ Then substituting for e from Eq (12) and performing the resulting integration, one obtains the total electric current from the last equation to be

$$i = \frac{I}{2\pi h_0 \eta \phi_0/B_0{}^2 r_0{}^2} = \frac{16(1-\zeta_1{}^2)}{(2\zeta_1 \ln\zeta_1)^2 + (8/M^2)(1-\zeta_1{}^2)\ln\zeta_1 - (\zeta_1{}^2-1)^2} \quad (25)$$

Substituting for i in Eq (24), one finally obtains

$$v = \frac{V_\theta}{\phi_0/B_0 r_0} = \mathcal{R}(\zeta_1,M)\left[c\left(\frac{1}{\zeta}-\zeta\right)+\zeta\ln\zeta\right] \quad (26)$$

in which

$$\mathcal{R}(\zeta_1,M) = \frac{4(1-\zeta_1{}^2)}{(2\zeta_1\ln\zeta_1)^2 + (8/M^2)(1-\zeta_1{}^2)\ln\zeta_1 - (\zeta_1{}^2-1)^2} \quad (27)$$

The pressure for ZRMF is found from Eqs (20, 23, and 24) to be

$$\pi = 2\mathcal{R}^2(\zeta_1,M)\int\left[c\left(\frac{1}{\zeta}-\zeta\right)+\zeta\ln\zeta\right]^2\frac{d\zeta}{\zeta} + A_2$$

This leads to the result

$$\frac{p_0 - p}{\rho\phi_0{}^2/2B_0{}^2r_0{}^2} = \mathcal{R}^2(\zeta_1,M)\left[c+\frac{1}{2}+\frac{c^2}{\zeta} - \left(c^2+c+\frac{1}{2}\right)\zeta^2 - 2c(\ln\zeta)^2 + 4c^2\ln\zeta + (1+2c)\zeta^2\ln\zeta - \zeta^2(\ln\zeta)^2\right] \quad (28)$$

if $p = p_0$ at $r = r_0$

By following the same procedure, it may be shown that, for the case of a field with nonzero radial mass flux (NRMF), the rotational velocity and pressure fields for viscous flow are given by

$$v = \frac{V_\theta}{\phi_0/B_0r_0} = \mathcal{R}(\zeta_1,M,R_N)\left[a_1\zeta^m + \frac{a_2}{\zeta} - \zeta\right] \quad (29)$$

$$\left.\begin{aligned}\frac{p_0-p}{\rho\phi_0{}^2/2B_0{}^2r_0{}^2} = \mathcal{R}^2(\zeta_1,M,R_N)\Big[(c_4{}^2+a_2{}^2)\left(\frac{1}{\zeta^2}-1\right) + \\ 1-\zeta^2+\frac{a_1{}^2}{m}(1-\zeta^{2m}) + \frac{4a_1a_2}{m-1}(1-\zeta^{m-1}) - \\ \frac{4a_1}{m+1}(1-\zeta^{m+1}) + 4a_2\ln\zeta\Big] \qquad \text{if } |R_u| \ll 1\S\end{aligned}\right\} \quad (30)$$

in which

$$\mathcal{R}(\zeta_1,M,R_N) = \frac{2(m+1)}{(m+1)\left\{\zeta_1{}^2 - 1 + 2\left[\frac{2R_N}{M^2}-a_2\right]\ln\zeta_1\right\} - 2a_1(\zeta_1{}^{m+1}-1)} \quad (31)$$

$$a_2 = \frac{\zeta_1{}^2 - \zeta_1{}^{m+1}}{1-\zeta_1{}^{m+1}} \qquad a_1 = 1 - a_2$$

$$m = R_N + 1 \qquad C_4 = \frac{\beta}{\mathcal{R}(\zeta_1,M,R_N)}$$

§ For a hydrogen plasma at 1 ev, with $R_N = -200$, $R_u = (\mu\sigma\eta/\rho)R_N = -2 \times 10^{-4}$

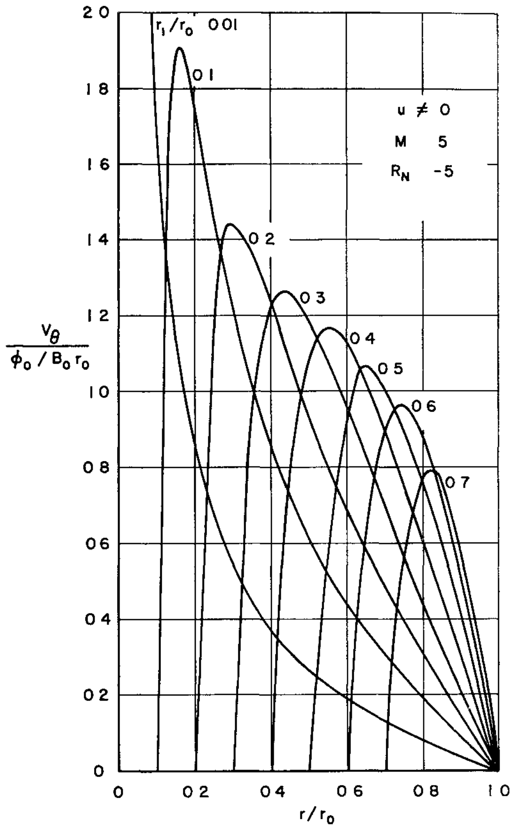

Fig 5 Variation of the rotational velocity of the viscous fluid with r_1/r_0

The total electric current is also found to be

$$\frac{I}{2\pi h_0\eta\phi_0/B_0{}^2r_0{}^2} = 4\mathcal{R}(\zeta_1,M,R_N)R_N \quad (32)$$

For a fluid with infinite electrical conductivity σ, the Hartmann number is infinite and consequently, setting $M = \infty$ in Eqs (25–32), one obtains the electric current input, the rotational velocity, and pressure fields Thus, the rotational velocity of a fluid with a very large electrical conductivity in a system without radial mass flow is found from Eqs (26) and (27) to be

$$\frac{V_\theta}{\phi_0/B_0r_0} = \mathcal{R}(\zeta_1)\left[c\left(\frac{1}{\zeta}-\zeta\right)+\zeta\ln\zeta\right] \quad (33)$$

where

$$\mathcal{R}(\zeta_1) = \frac{4(1-\zeta_1{}^2)}{(2\zeta_1\ln\zeta_1)^2 - (\zeta_1{}^2-1)^2} \quad (34)$$

For an incompressible inviscid fluid it can be shown[9] from Eqs (3–5) that the azimuthal velocity is given by

$$\frac{V_\theta}{\phi_0/B_0r_0} = \frac{1}{\zeta\ln(\zeta_1)} \qquad \text{if} \qquad G_0 = 0$$

and

$$\frac{V_\theta}{\phi_0/B_0r_0} = \frac{1}{\ln\zeta}\left\{\frac{1}{\zeta}\left[1 - \beta_1\ln\zeta_1 + \frac{\beta_2}{4}(1-\zeta_1{}^2)\right] + \frac{1}{2}\beta_2\zeta\ln\zeta_1\right\} \text{ if } G_0 \neq 0$$

From Eq (4), if G_0 and η are zero, the electric current is found to be zero and consequently B_θ, the azimuthal induced magnetic field, must be zero for ZRMF [see Eq (18)] From Eq (5), B_θ is found to be inversely proportional to r for the inviscid fluid if G_θ is not zero (NRMF) With these expressions for V_θ and B_θ, the pressure field of an inviscid fluid may be determined from Eq (20) for either ZRMF or NRMF

The variation with radius of the azimuthal velocity and pressure profiles with the parameters are shown in Figs 3–8. For purposes of comparison, plots of the velocity and pressure of the inviscid fluid are superimposed in dotted broken lines on those for the corresponding cases of the viscous fluid. To distinguish between the cases considered, the graphs for the inviscid fluid are labelled $M = \infty$ or $\eta = 0$ and those for ZRMF and NRMF are labeled $u = 0$ and $u \neq 0$, respectively

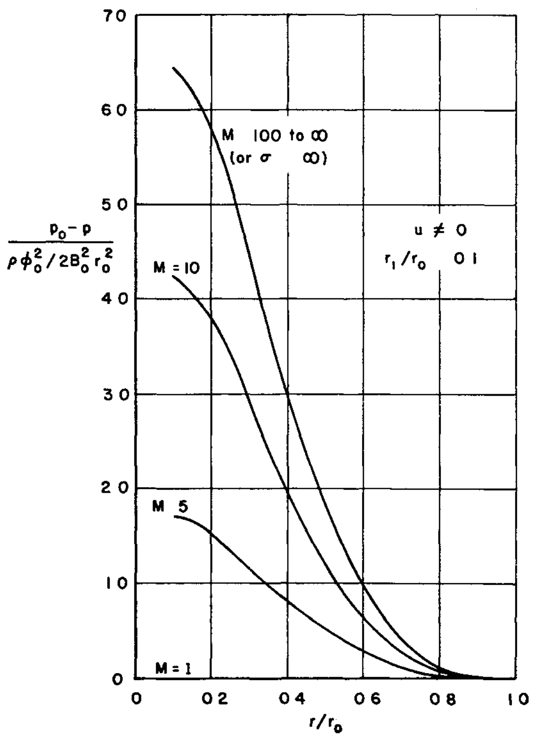

Fig 6 Variation of the pressure field of the viscous fluid with *M*

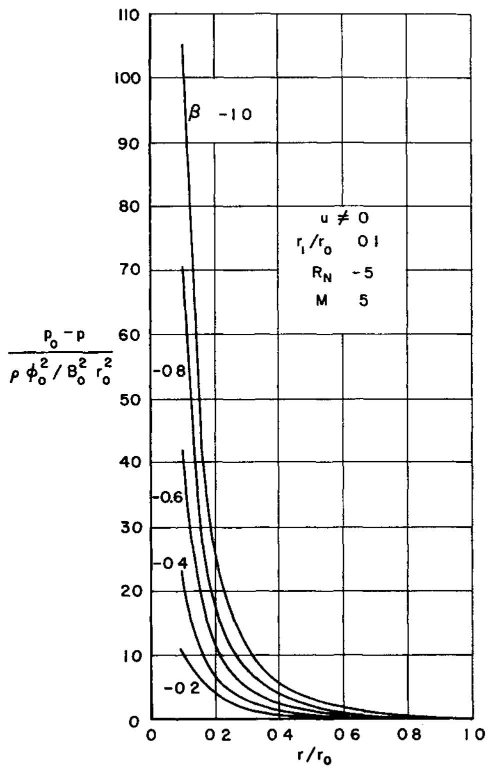

Fig 8 Variation of the pressure field of the viscous fluid with β

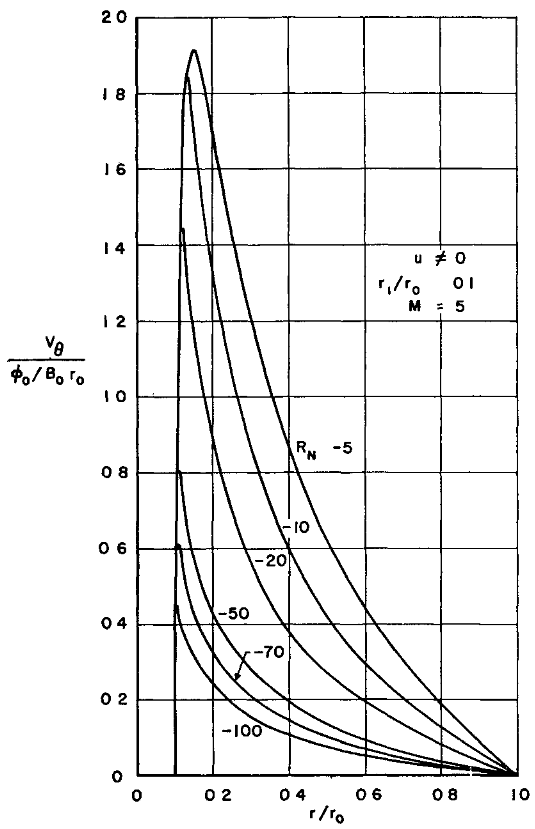

Fig 7 Variation of the pressure field of the viscous fluid with r_1/r_0

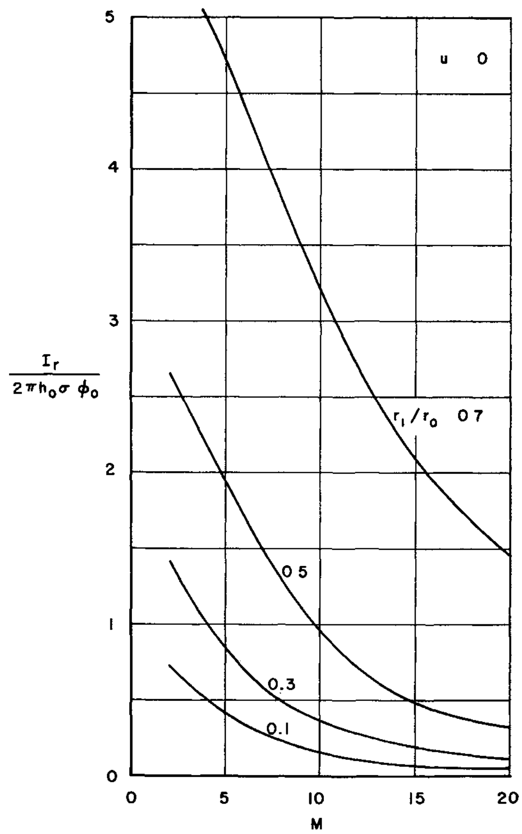

Fig 9 Variation of the absolute radial electric current with r_1/r_0 vs *M*

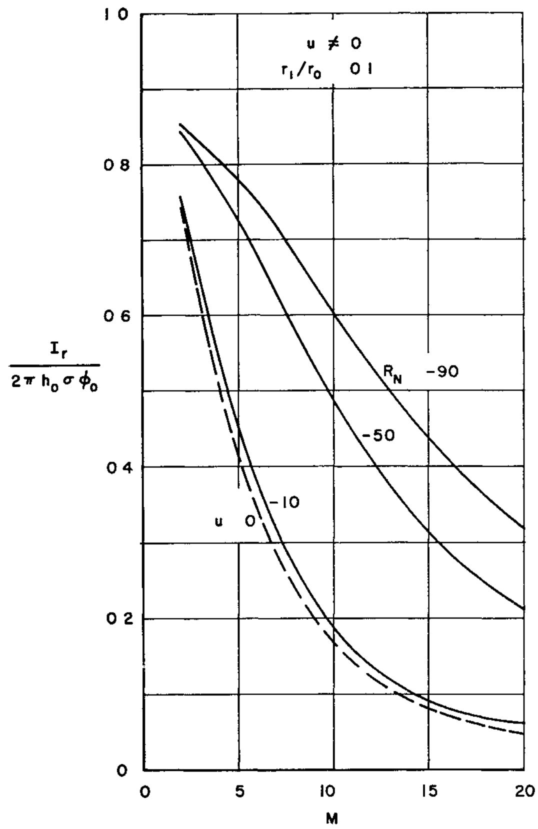

Fig 10 Variation of the absolute ıadial electric current with R_N vs M

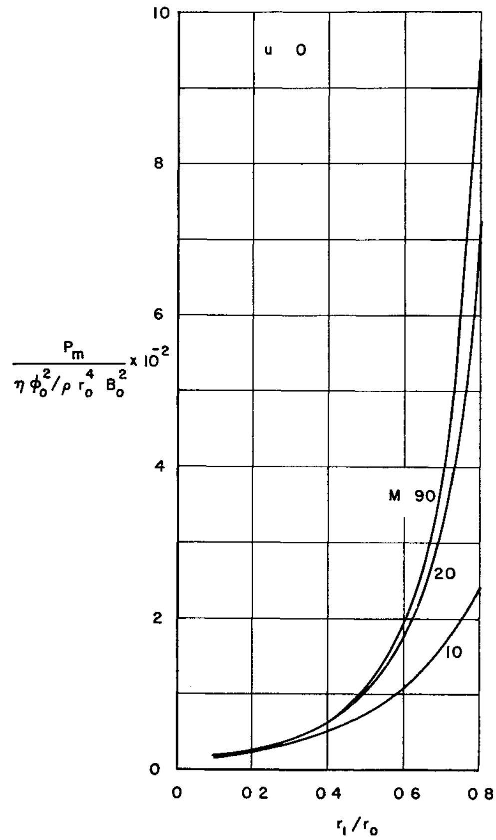

Fig 11 Variation of electrical power per unit mass with M

From Figs 3 and 4 it is obvious that there is no asymptotic agreement between the solutions for the inviscid fluid and those for the viscous fluid The reason for this is the fact that if one sets the coefficient of viscosity equal to zero in the differential equation, one obtains a differential equation of the first order requiring only one boundary condition on V_θ as distinct from the viscous fluid analysis for which the differential equation is of the second order

It is interesting to note from Figs 3 and 4 that the azimuthal velocity of the viscous fluid increases locally with increasing M, the Hartmann number An increase of M may be interpreted as a decrease of η; therefore, the local azimuthal velocity is higher for a less viscous fluid as expected The effect of viscosity is exemplified in Fig 5 Figure 5 shows that the local rotational velocity decreases with increasing r_1/r_0, owing to radial shear effects on the velocity field

4 Further Macroscopic Quantities

Quantities of considerable interest in the application of the rotating plasma include the electrical power input, total rotational kinetic energy, and the dissipative mechanisms of Joule heating and viscous dissipation

The electrical power required to rotate the plasma per unit mass is expressed by

$$P_m = \frac{|I_r|\phi_0}{4\pi h_0(r_0^2 - r_1^2)\rho}$$

where $|I\ |$ stands for the absolute magnitude of I Substituting for I_r from Eqs (25) and (32), it is found that, for ZRMF,

$$\frac{P_m}{\eta\phi_0^2/\rho B_0^2 r_0^4} = \frac{4\,|\mathcal{R}(\zeta_1,M)|}{1 - \zeta_1^2} \qquad (35)$$

and for NRMF,

$$\frac{P_m}{\eta\phi_0^2/\rho B_0^2 r_0^4} = \frac{4\,|R_N\mathcal{R}(\zeta,M,R_N)|}{1 - \zeta_1^2} \qquad (36)$$

n which $\mathcal{R}(\zeta_1,M)$ and $\mathcal{R}(\zeta_1,M,R_N)$ are given in Eqs (27) and (31) Here again, for a fluid with infinite electrical conductivity one sets $M = \infty$ in $\mathcal{R}(\zeta_1,M)$ and $\mathcal{R}(\zeta_1,M,R_N)$ and uses Eqs (34) and (35) to obtain the power requirements In Eq (35) it is necessary to take the absolute value of R_N since it may be negative or positive depending upon whether radial mass flow is inward or outward In this analysis, the source of radial mass flow is assumed to be at $r = r_0$ while the sink is at $r = r_1$, and so R_N is taken to be negative

The absolute magnitude of the radial electric current I and the electrical power input per total mass of plasma in the system P_m are plotted in Figs 9–11 Figures 9 and 10 clearly show that the absolute magnitude of the dimensionless electric current decreases with increasing M, negatively decreasing R_N, and decreasing r_1/r_0 It also may be noted that the radial electric current either approaches the limiting value of zero or remains constant for large values of M This means that the radial electric current of the viscous fluid becomes less for a less viscous fluid until, in the limiting case of an inviscid fluid for which $M = \infty$, the current becomes zero or very small according as radial mass flow is zero or nonzero For steady-state rotation of the viscous fluid, the viscous forces are balanced by the Lorentz force, which results from the radial electric current For a fluid with very small coefficient of viscosity or very large Hartmann number, this analysis shows that the Lorentz force or equivalently the radial electric current required for rotation is correspondingly small, as one would expect For the inviscid fluid for which, by definition, there are no viscous forces, the radial electric current is correspondingly zero if there is no radial mass flow As seen from the influence of R_N (which

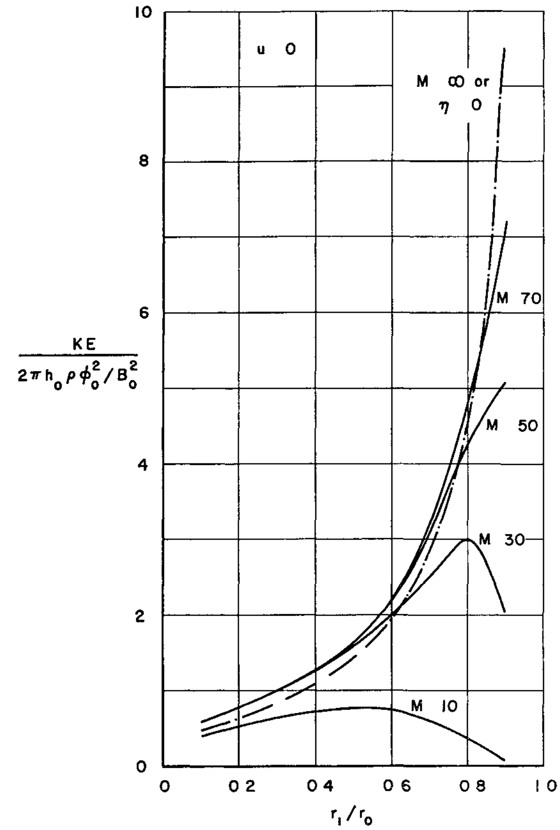

Fig 12 Variation of total rotational kinetic energy with with M vs r_1/r_0

indicates the effect of G_0, the radial mass flow rate at $r = r_0$) on I_r, as depicted in Fig 10, large radial mass flow induces large current flow since more electrical power is required to produce more radial mass flow

The rotational kinetic energy stored in this device as a result of the electrical energy input is given by

$$\mathrm{KE} = 2\pi \int_{-h_0}^{h_0} \int_{r_1}^{r_0} \frac{\rho V_\theta^2}{2}\, r dz dr$$

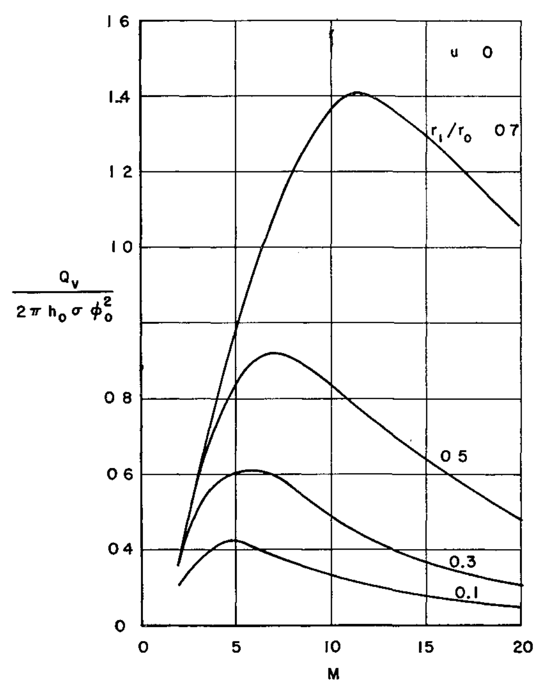

Fig 13 Variation of viscous dissipation with r_1/r_0 vs M

Hence, for an incompressible fluid whose rotational velocity is independent of z,

$$\mathrm{KE} = 2\pi h_0 r_0^2 \rho \int_{\zeta_1}^{1} V_\theta^2 \zeta d\zeta$$

By substituting for V_θ and performing the resulting integration, the total rotational kinetic energy stored in a system without radial mass flow is

$$\frac{\mathrm{KE}}{2\pi h_0 \rho \phi_0^2/B_0^2} = \mathcal{R}^2(\zeta_1,M) \left\{ c^2\left[\frac{1}{4}(1-\zeta_1^4) - \ln\zeta_1 + \zeta_1^2 - 1\right] - \frac{\zeta_1^4}{8}[2(\ln\zeta_1)^2 - \ln\zeta_1] + c\left[\frac{1}{8}\left(1-\zeta_1^4\right) + \frac{\zeta_1^4}{2}\ln\zeta_1 + \frac{1}{2}(\zeta_1^2 - 1) - \zeta_1^2\ln\zeta_1\right] + \frac{1}{32}(1-\zeta_1^4)\right\} \quad (37)$$

and the total rotational kinetic energy stored in a system

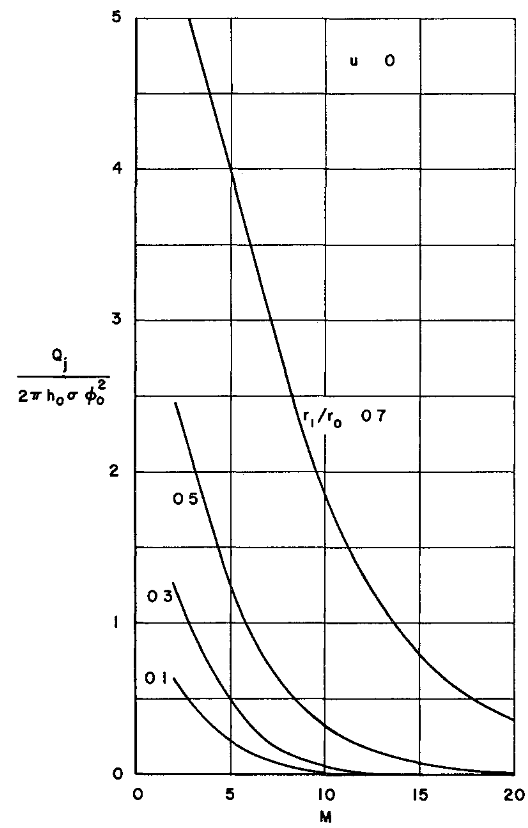

Fig 14 Variation of Joule heating with r_1/r_0 vs M

with radial mass flux is given by

$$\frac{\mathrm{KE}}{2\pi h_0 \rho \phi_0^2/B_0^2} = \mathcal{R}^2(\zeta_1,M,R_N) \left\{ \frac{a_1^2}{2(m+1)}[1 - \zeta_1^{2m+2}] - a_2\ln\zeta_1 + \frac{1}{4}(1-\zeta_1^4) + \frac{2a_1a_2}{m+1}(1-\zeta_1^{m+1}) - \frac{2a_1}{m+3}(1-\zeta_1^{m+3}) - a_2(1-\zeta_1^2)\right\} \quad (38)$$

A typical plot of the rotational kinetic energy is shown in Fig 12

Finally, under steady conditions, the electrical power input will be dissipated into heat as a result of Joule heating and viscous dissipation The total rate of Joule heating is the integral of J_θ^2/σ over the whole volume of the device and it is given by

$$Q_j = \frac{2\pi}{\sigma}\int_{z=-h_0}^{h_0}\int_{r=r_1}^{r_0}\left[\frac{I}{4\pi h_0 r}\right]^2 rdrdz$$
$$= \frac{I^2}{4\pi h_0\sigma}\ln\left(\frac{1}{\zeta_1}\right) \quad (39)$$

with I_r given by Eqs (25) and (32) for ZRMF and NRMF, respectively Regrouping the terms of the resulting equations, one obtains the expressions

$$\frac{Q_j}{2\pi h_0\sigma\phi_0^2} = \mathcal{R}^2(\zeta,M)\,\frac{8}{M^4}\ln\left(\frac{1}{\zeta_1}\right) \text{ for ZRMF} \quad (40)$$

$$\frac{Q_j}{2\pi h_0\sigma\phi_0^2} = \mathcal{R}^2(\zeta,M,R_N)\,\frac{8R_N^2}{M^4}\ln\left(\frac{1}{\zeta_1}\right) \text{ for NRMF} \quad (41)$$

The rate of viscous dissipation is also given by the integral

$$Q_V = \iiint \eta\phi_V d_V = 4\pi h_0 r_0^2\,\eta\int_{\zeta_1}^{1}\zeta\phi_V d\zeta$$

in which

$$\phi_V = 2\left[\left(\frac{du}{dr}\right)^2 + \left(\frac{u}{r}\right)^2\right] + \left[r\frac{d}{dr}\left(\frac{V_\theta}{r}\right)\right]^2 - \frac{2}{3}\left[\frac{1}{r}\frac{d}{dr}(ru)\right]^2$$

Hence, for the system under consideration, a rearrangement of terms yields the expression for the rate of viscous dissipation as

$$\frac{Q_V}{2\pi h_0\sigma\phi_0^2} = \frac{1}{M^2}\int_{\zeta_1}^{1}\left\{\frac{\beta^2}{\zeta^4} + \mathcal{R}^2\left[\zeta\frac{d}{d\zeta}\left(\frac{v}{\zeta}\right)\right]^2\right\}\zeta d\zeta$$

Substituting for v from Eqs (24) and (29) into the last equation and performing the indicated integration, one obtains for the rate of viscous dissipation for a system without radial mass flux as

$$\frac{Q_V}{2\pi h_0\sigma\phi_0^2} = \frac{\mathcal{R}^2(\zeta_1,M)}{M^2}\left[\frac{(1-\zeta_1^2)^2 - (2\zeta_1\ln\zeta_1)^2}{1-\zeta_1^2}\right] \quad (42)$$

and for a system with radial mass flux,

$$\frac{Q_V}{2\pi h_0\sigma\phi_0^2} = \frac{\mathcal{R}^2(\zeta_1,M,R_N)}{M^2}\left\{\left[\frac{4}{\zeta_1^2} - 4\right]\times\left[\frac{\beta^2}{\mathcal{R}^2(\zeta_1,M,R_N)} + a_2^2\right] + 2\left[\frac{a_1^2R_N^2}{2m}(1-\zeta_1^{2m}) - 4a_1a_2(1-\zeta_1^{m-1})\right]\right\} \quad (43)$$

With reference to Fig 11, it is observed that the electric power per unit mass of plasma rapidly increases as r_1/r_0 approaches unity and also that it increases asymptotically with increasing M However, it may be recalled that the rotational velocity considerably decreases as r_1/r_0 approaches unity Hence, in spite of the large amount of electrical power required for the rotating plasma as the ratio of the inner to the outer radii approaches unity, it is found that most of the energy input is dissipated into heat by viscous dissipation and Joule heating Figures 13 and 14 illustrate the fact that the available energy is largely converted into heat by the viscous mixing and Joule heating of the plasma as r_1/r_0 approaches unity

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